§5.6 Problems: extremal graph theory

There are many good problems in the book. I recommend for this set of notes you look primarily at the exercises from Section 7.3 and at the ones from Sections 11.1 and 11.2 for practice problems. Below, you have some additional extremal graph theory problems. Give them a try if you have time! Some of them will be discussed by your TA during the upcoming discussion sessions from 4 to 5 PM on Tuesdays and Thursdays.

Problem 5.6.1. Prove that a graph with n vertices and k edges has at least $\frac{k}{3n}(4k-n^2)$ triangles.

Problem 5.6.2. Prove Zarankiewicz's Lemma: In any graph G with no K_r subgraph, prove that there exists a vertex with degree at most $\lfloor \frac{r-2}{r-1}n \rfloor$.

Problem 5.6.3. Use Zarankiewicz's Lemma to give another proof of Turan's theorem.

Problem 5.6.4. For a pair $A = (x_1, y_1)$ and $B = (x_2, y_2)$ of points in the plane, let

$$d(A, B) = |x_1 - x_2| + |y_1 - y_2|.$$

We call a pair (A, B) of (unordered) points harmonic if $1 < d(A, B) \leq 2$. Determine the maximum number of harmonic pairs among 100 points in the plane.

Problem 5.6.5. A graph G has $n^2 + 1$ edges and $2n$ vertices.
a) Prove that it contains two triangles sharing a common edge.
b) Prove that it contains at least n triangles.

Problem 5.6.6. A graph G has n vertices and no triangles. No matter how we partition its vertices into two classes, there are two adjacent vertices in the same class. Prove that some vertex has degree at most $2n/5$.

Problem 5.6.7. The edges of the complete graph on $2^n + 1$ vertices are colored in one of n colors. Prove that there is a monochromatic cycle of odd length.

Problem 5.6.8. We have n charged batteries, n uncharged batteries and a radio which needs two charged batteries to work. Suppose we don't know which batteries are charged and which ones are uncharged. Find the least number of attempts sufficient to make sure the radio will work. An attempt consists in putting two batteries in the radio and check if the radio works or not.

Problem 5.6.9. Given a graph with $2n + 1$ vertices, such that for any n vertices, there exists another one connected to all these n vertices. Show that there is a vertex connected to all the other $2n$ vertices.

Problem 5.6.10. A graph G has n vertices and $nk/2$ edges, with $k \geq 1$. Prove that the maximum number of pairwise nonadjacent vertices of G is at least $\frac{n}{k+1}$.

Problem 5.6.11. A graph G with $2n$ vertices is given. For every n different vertices of G, there exists a vertex connected with all of them. What is the minimal possible number of edges of G ?

Problem 5.6.12. The vertices of a finite connected graph cannot be colored with less than $n + 1$ colors so that adjacent vertices have different colors. Prove that $\frac{n(n-1)}{2}$ edges can be removed from the graph so that it remains connected.